Problem Solutions

SOLUTIONS FOR CHAPTER 1

1.1 Color JPEG compressed images are typically 5 to 50 times smaller than they would be if stored "naively", so the ratio of naively-stored to JPEG-stored might range from a low of 0.02 to 0.2.

1.2

- From Euler we have (for x real)

$$\begin{aligned} e^{-ix} &= \cos(x) + i\sin(-x) \\ &= \cos(x) - i\sin(x) \\ &= \overline{\cos(x) + i\sin(x)} \\ &= \overline{e^{ix}}. \end{aligned}$$

- If $e^{2\pi ix} = 1$ then $\cos(2\pi x) + i\sin(2\pi x) = 1$. We conclude that $\sin(2\pi x) = 0$, which forces $2\pi x = 2\pi k$ for some integer k, i.e., $x = k$. Conversely, if x is an integer k then $e^{2\pi ix} = e^{2\pi ik} = \cos(2\pi k) + i\sin(2\pi k) = 1$.

1.3 The eighth roots of unity are the numbers $e^{2\pi ik/8} = \cos(k\pi/4) + i\sin(k\pi/4)$ where $0 \leq k \leq 7$, and are equal to

$$1, \ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \ i, \ -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \ -1, \ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \ -i, \ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

(moving counterclockwise around the unit circle).

1.4 The mth component of $\mathbf{E}_{N,k}$ is given by equation (1.23) and is $\mathbf{E}_{N,k}(m) = e^{2\pi ikm/N}$. But $(e^{2\pi ikm/N})^N = e^{2\pi ikm} = 1$ since k and m are integers (refer to Exercise 1.2).

1.5 We have from Euler's identity

$$\begin{aligned} x(t) = a\cos(\omega t) + b\sin(\omega t) &= \frac{a}{2}(e^{i\omega t}) + e^{-i\omega t}) + \frac{b}{2i}(e^{i\omega t}) - e^{-i\omega t}) \\ &= \frac{a}{2}(e^{i\omega t} + e^{-i\omega t}) - \frac{ib}{2}(e^{i\omega t}) - e^{-i\omega t}) \\ &= \frac{1}{2}(a - ib)e^{i\omega t} + \frac{1}{2}(a + ib)e^{-i\omega t} \end{aligned}$$

(since $1/i = -i$). Comparison to $x(t) = ce^{i\omega t} + de^{-i\omega t}$ shows that $c = \frac{a-ib}{2}, d = \frac{a+ib}{2}$. These last two linear equations are easily solved for a and b to yield $a = c + d, b = i(c - d)$.

1.6

a. The sampled versions are

$$\begin{aligned} \mathbf{x} &= < 0, 0.325, 0.65, 0.975 > \\ \mathbf{y} &\approx < 0, 0.382, 0.707, 0.924 > \end{aligned}$$

The result of sampling $x(t) + y(t)$ is just $\mathbf{x} + \mathbf{y} \approx < 0, 0.708, 1.36, 1.90 >$.

b. We find

$$\begin{aligned} q(\mathbf{x}) &= < 0, 0, 1, 1 > \\ q(\mathbf{y}) &= < 0, 0, 1, 1 > \\ q(\mathbf{x} + \mathbf{y}) &= < 0, 1, 1, 2 > . \end{aligned}$$

Then $q(\mathbf{x}) + q(\mathbf{y}) \neq q(\mathbf{x} + \mathbf{y})$. Also, $q(2\mathbf{x}) =< 0, 1, 1, 2 >$ which is not $2q(\mathbf{x})$.

1.7 Yes, this is a vector space. It's clearly closed under addition and scalar multiplication; note that we consider something like $a_0 + a_1x + 0x^2$ to be a quadratic polynomial. Addition commutes, is associative, the zero vector is the zero polynomial, and all the other rules of Definition 1.4.1 are straightforward to verify.

1.8 This is not a vector space—it's not closed under addition or scalar multiplication, e.g., if

$$\int_0^1 f(x)\,dx = 3$$

then the function $2f$ satisfies

$$\int_0^1 2f(x)\,dx = 6$$

and isn't in the set.

1.9 $\mathbb{R}^n$ is not a subspace of $\mathbb{C}^n$, if $\mathbb{C}^n$ is taken as a vector space over $\mathbb{C}$, for we don't have closure under scalar multiplication. For example, in the case $n = 2$ we can take $\mathbf{v} =< 1, 2 >\in \mathbb{R}^2$ and scalar $c = i$. But $c\mathbf{v} =< i, 2i >$ is not in $\mathbb{R}^2$.

1.10 With the operations as defined we clearly have closure in addition and scalar multiplication. The commutativity and associativity of addition follow from the same properties for the reals numbers. The zero vector is just the element

$$\mathbf{0} = (0, 0, 0, \ldots)$$

The inverse of any element $\mathbf{x} = (x_0, x_1, \ldots)$ is

$$(-x_0, -x_1, -x_2, \ldots)$$

Properties (e)-(h) in Definition 1.4.1 follow immediately from application of the corresponding properties for real numbers.

1.11 As remarked in the text (Example 1.8) we have $(p+q)^2 \le 2p^2 + 2q^2$ for any real numbers p and q. As a result if $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ are elements of $L^2(\mathbb{N})$ then $(x_k + y_k)^2 \le 2x_k^2 + 2y_k^2$. Sum both sides to conclude that

$$\sum_{k=0}^{\infty}(x_k + y_k)^2 \le 2\left(\sum_{k=0}^{\infty} x_k^2 + \sum_{k=0}^{\infty} y_k^2\right) < \infty$$

so the sum $\mathbf{x}+\mathbf{y}$ is in $L^2(\mathbb{N})$ and we have closure under addition. Clearly $\sum_k (cx_k)^2 = c^2 \sum_k x_k^2$, so we have closure under scalar multiplication. All other properties are verified exactly just as for Exercise 1.10.

If $\sum_k x_k^2 < \infty$ then from $|x_m|^2 \le \sum_k x_k^2$ for all m we immediately see that $|x_m| \le M = \left(\sum_k x_k^2\right)^{1/2}$ for all $m \ge 0$. That is, all x_m satisfy the common bound $|x_m| \le M$, so $L^2(\mathbb{N})$ is a subset of $L^\infty(\mathbb{N})$ which is closed under addition and scalar multiplication, hence a subspace.

1.12

a. If there were two "zero vectors" $\mathbf{0}_1$ and $\mathbf{0}_2$ then we must have $\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_1$ from property (3/c) in Definition 1.4.1. But we also must have $\mathbf{0}_1 + \mathbf{0}_2 = \mathbf{0}_2$. Comparison of the last two equations shows that $\mathbf{0}_1 = \mathbf{0}_2$.

b. That the left and right sides in line 1 are equal follows from (3/g). The left side of line 2 equals the left side of 1 since $1 + 0 = 1$ in the reals, and the right side of line 2 equals the right side of line 1 due to property (3/h). The left side of line 3 equals the left side of line 2 due to (3/h). Line 4 follows by adding $-u$ to both sides of line 3 and invoking commutativity and associativity (3/a, 3/b). Line 5 follows from the definition of the additive inverse $-u$, and line 6 follows from the definition of $\mathbf{0}$, the zero vector.

c. Many ways to proceed. Start with $\mathbf{u} + \mathbf{v} = \mathbf{0}$, which is equivalent to

$$\mathbf{u} + \mathbf{v} = (1 + (-1))\mathbf{u}$$

since from part (b) we know $0\mathbf{u} = \mathbf{0}$ (and $1 + (-1) = 0$ in the reals!) The above displayed equation leads to (distribute according to property (3/f)) $\mathbf{u} + \mathbf{v} = \mathbf{u} + (-1)\mathbf{u}$. Add $-\mathbf{u}$ to both sides and get the desired conclusion, $\mathbf{v} = (-1)\mathbf{u}$.

1.13 The vectors are, for the case $N = 2$,

$$\mathbf{E}_{2,-2} = \mathbf{E}_{2,0} = \mathbf{E}_{2,2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{E}_{2,-1} = \mathbf{E}_{2,1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The vectors are, for the case $N = 3$,

$$\mathbf{E}_{3,-2} = \mathbf{E}_{3,1} = \begin{bmatrix} 1 \\ \frac{-1+i\sqrt{3}}{2} \\ \frac{-1-i\sqrt{3}}{2} \end{bmatrix}, \; \mathbf{E}_{3,-1} = \mathbf{E}_{3,2} = \begin{bmatrix} 1 \\ \frac{-1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{bmatrix}, \; \mathbf{E}_{3,0} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

1.14

a. We have $|f(t)| = |a||e^{i\omega t}| = |a|$ (since it follows from Euler's identity that $|e^{i\omega t}| = \sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = 1$). By the same reasoning we have $|a| = |r||e^{i\theta}| = |r| = r$ (since $r > 0$). Then $|f(t)| = r$.

b. From $f(t) = ae^{i\omega t}$ and $a = re^{i\theta}$ we have

$$f(t) = re^{i(\omega t + \theta)} = r\cos(\omega t + \theta) + ir\sin(\omega t + \theta).$$

Now, for example,

$$\cos(\omega t + \theta) = \cos(\omega(t + \theta/\omega))$$

so that the graph of $\cos(\omega t + \theta)$ is the graph of $\cos(\omega t)$ shifted a distance θ/ω to the left. This corresponds to a fraction θ/ω of one period. A similar consideration applies to $\sin(\omega t + \theta)$. We conclude that the graph of $ae^{i\omega t}$ is just $e^{i\omega t}$ shifted "to the left". This actually corresponds to advancing the signal in time (think about why—$f(t)$ will attain a given value θ/ω in advance of $e^{i\omega t}$).

1.15 We can write (use Euler's identity)

$$\begin{aligned}\cos(\alpha x)\cos(\beta y) &= \frac{e^{i\alpha x}+e^{-i\alpha x}}{2}\frac{e^{i\beta y}+e^{-i\beta y}}{2}\\ &= \frac{1}{4}e^{i\alpha x}e^{i\beta y}+\frac{1}{4}e^{-i\alpha x}e^{i\beta y}+\frac{1}{4}e^{i\alpha x}e^{-i\beta y}+\frac{1}{4}e^{-i\alpha x}e^{-i\beta y}.\end{aligned}$$

Similarly

$$\begin{aligned}\cos(\alpha x)\sin(\beta y) &= -\frac{i}{4}e^{i\alpha x}e^{i\beta y}-\frac{i}{4}e^{-i\alpha x}e^{i\beta y}+\frac{i}{4}e^{i\alpha x}e^{-i\beta y}+\frac{i}{4}e^{-i\alpha x}e^{-i\beta y}\\ \sin(\alpha x)\cos(\beta y) &= -\frac{i}{4}e^{i\alpha x}e^{i\beta y}+\frac{i}{4}e^{-i\alpha x}e^{i\beta y}-\frac{i}{4}e^{i\alpha x}e^{-i\beta y}+\frac{i}{4}e^{-i\alpha x}e^{-i\beta y}\\ \sin(\alpha x)\sin(\beta y) &= -\frac{1}{4}e^{i\alpha x}e^{i\beta y}+\frac{1}{4}e^{-i\alpha x}e^{i\beta y}+\frac{1}{4}e^{i\alpha x}e^{-i\beta y}-\frac{1}{4}e^{-i\alpha x}e^{-i\beta y}\end{aligned}$$

1.16 The relation $\mathbf{E}_k = \mathbf{C}_k + i\mathbf{S}_k$ follows from Euler's identity, for the mth component of $\mathbf{E}_k$ is $\mathbf{E}_k(m) = e^{2\pi imk/N}$, which is $\cos(2\pi mk/N) + i\sin(2\pi mk/N) = \mathbf{C}_k(m) + i\mathbf{S}_k(m)$, where $\mathbf{C}_k(m)$ and $\mathbf{S}_k(m)$ are the mth components of $\mathbf{C}_k$ and $\mathbf{S}_k$, respectively.

The equation $\overline{\mathbf{E}_k} = \mathbf{C}_k - i\mathbf{S}_k$ follows from $\overline{\mathbf{E}_k(m)} = \overline{e^{2\pi imk/N}} = e^{-2\pi imk/N} = \cos(2\pi mk/N) - i\sin(2\pi mk/N) = \mathbf{C}_k(m) - i\mathbf{S}_k(m)$. The relations $\mathbf{C}_k = \frac{1}{2}(\mathbf{E}_k + \overline{\mathbf{E}_k})$ and $\mathbf{S}_k = \frac{1}{2i}(\mathbf{E}_k - \overline{E_k})$ are similar and follow from component-by-component application of $\mathbf{C}_k(m) = \frac{1}{2}(\mathbf{E}_k(m) + \overline{\mathbf{E}_k(m)}$ and $\mathbf{C}_k(m) = \frac{1}{2i}(\mathbf{E}_k(m) - \overline{\mathbf{E}_k(m)}$ (which themselves are consequences of equation (1.13)).

The final two relations are similar and consequences of component-by-component application of $\mathbf{C}_k(m) = \mathrm{Re}(\mathbf{E}_k(m))$ and $\mathbf{S}_k(m) = \mathrm{Im}(\mathbf{E}_k(m))$.

1.17 If we treat $\mathbf{E}_{m,k}$ as an $m \times 1$ matrix, the row ith column r entry is

$$\mathbf{E}_{m,k}(r,1) = e^{2\pi irk/m}.$$

The quantity $\mathbf{E}_{n,l}^T$ is a $1 \times n$ matrix with row 1 column s entry

$$\mathbf{E}_{n,l}^T(1,s) = e^{2\pi isl/n}.$$

The definition of matrix multiplication dictates that the row r column s entry of the product $\mathbf{E}_{m,k}\mathbf{E}_{n,l}^T$ is given by

$$\begin{aligned}(\mathbf{E}_{m,k}\mathbf{E}_{n,l}^T)(r,s) &= \sum_{p=1}^{1}(\mathbf{E}_{m,k}(r,p)(\mathbf{E}_{n,l}^T)(p,s)\\ &= (\mathbf{E}_{m,k}(r,1)(\mathbf{E}_{n,l}^T)(1,s)\\ &= e^{2\pi irk/m}e^{2\pi isl/n}\\ &= e^{2\pi i(rk/m+sl/n)}.\end{aligned}$$