

## SOLUTIONS TO PROBLEMS 2

2.1 From (2.8),

$$x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \left(\frac{1}{\alpha\beta}\right) = 0, \text{ i.e. } \alpha\beta x^2 - (\alpha + \beta)x + 1 = 0.$$

But  $\alpha + \beta = 2$  and  $\alpha\beta = -3$ , so the required equation is  $3x^2 + 2x - 1 = 0$ .

2.2 Rearranging the expression for  $p$ , we have the equation

$$5x^2 - 3px + (p + 10) = 0.$$

Then using the general formula for the solution of a quadratic equation, (2.6b), real roots are only possible if  $9p^2 \geq 20(p + 10)$ .

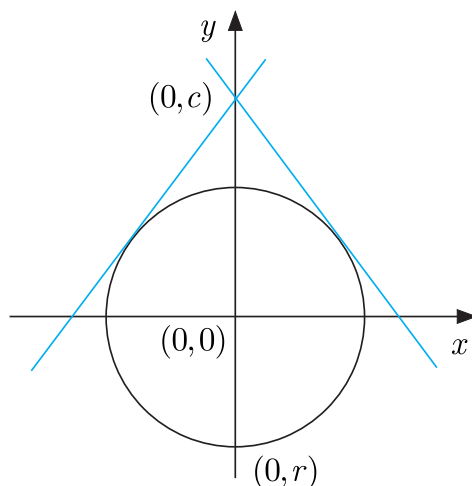
2.3 We need to solve the equation  $y = mx + c$  simultaneously with the equation for the circle. Substituting for  $y$  gives, after rearrangement,

$$(1 + m^2)x^2 + 2mcx + (c^2 - r^2) = 0,$$

which is a quadratic of the form  $\alpha x^2 + \beta x + \gamma = 0$ . This will have a single real root if  $\beta^2 = 4\alpha\gamma$ , i.e.

$$4m^2c^2 = 4(1 + m^2)(c^2 - r^2), \quad \text{or} \quad m = \pm \left(\frac{c^2 - r^2}{r^2}\right)^{1/2}.$$

Thus there are two tangents as shown in the figure below.



2.4 The equations of the two circles are

$$(x - 1)^2 + (y + 1)^2 = 4, \quad \text{i.e. } x^2 + y^2 - 2x + 2y - 2 = 0, \quad (1)$$

and

$$x^2 + y^2 = 4, \quad (2)$$

respectively. Subtracting (2) from (1) gives

$$-2x + 2y = -2, \quad \text{i.e. } y = x - 1 \quad (3)$$

and substituting in (2) gives  $2x^2 - 2x - 3 = 0$ , with solutions

$$x = (1 + \sqrt{7})/2 \quad \text{and} \quad x = (1 - \sqrt{7})/2$$

respectively. From (3) the corresponding values of  $y$  are

$$y = (-1 + \sqrt{7})/2 \quad \text{and} \quad y = (-1 - \sqrt{7})/2.$$

Hence the co-ordinates of the points of intersection are

$$(x, y) = \left(\frac{1}{2} + \frac{\sqrt{7}}{2}, -\frac{1}{2} + \frac{\sqrt{7}}{2}\right) \quad \text{and} \quad \left(\frac{1}{2} - \frac{\sqrt{7}}{2}, -\frac{1}{2} - \frac{\sqrt{7}}{2}\right).$$

The length of the chord is  $\sqrt{14}$  and the radius of the circle is 2 in each case. Hence the cosine rule (2.40a), with  $a = \sqrt{14}$ ,  $b = c = 2$ , gives  $\cos A = -6/8$  and  $A = 2.42 \text{ rad} = 139^\circ$ .

2.5 (a) We have

$$(x^3 + x^2 - x - 4) = (x - 1)(ax^2 + bx + c) + R(x).$$

Setting  $x = 1$  gives  $R = -3$ ; and multiplying out the bracket and equating powers of  $x$  gives  $a = 1$ ,  $b = 2$ ,  $c = 1$ , so that

$$(x^3 + x^2 - x - 4) = (x - 1)(x^2 + 2x + 1) - 3.$$

(b) By long division,

$$\begin{array}{r} 3x^2 + 5x + 5 \\ (x^2 - 2x + 3) \overline{) 3x^4 - x^3 + 4x^2 + 5x + 15} \\ \underline{3x^4 - 6x^3 + 9x^2} \phantom{+ 5x + 15} \\ 5x^3 - 5x^2 + 5x + 15 \\ \underline{5x^3 - 10x^2 + 15x} \phantom{+ 15} \\ 5x^2 - 10x + 15 \\ \underline{5x^2 - 10x + 15} \\ 0 \end{array}$$

so that

$$3x^4 - x^3 + 4x^2 + 5x + 15 = (x^2 - 2x + 3)(3x^2 + 5x + 5),$$

with the remainder  $R(x) = 0$ . Since both  $(x^2 - 2x + 3)$  and  $(3x^2 + 5x + 5)$  are of the form  $ax^2 + bx + c$  with  $b^2 < 4ac$ , both of them, and hence the quartic  $f(x)$  itself, have no real roots.

2.6 By inspection, one finds that  $x = 1$  and  $x = 2$  are roots. Hence, by the factor theorem,

$$x^4 - 2x^3 - 2x^2 + 5x - 2 = (x - 1)(x - 2)(ax^2 + bx + c).$$

By comparing powers of  $x^4$  on both sides one finds  $a = 1$ , and by comparing the constant term,  $c = -1$ . Hence

$$x^4 - 2x^3 - 2x^2 + 5x - 2 = (x - 1)(x - 2)(x^2 + bx - 1).$$

Also, comparing powers of  $x^3$  gives  $b = 1$  and hence

$$x^4 - 2x^3 - 2x^2 + 5x - 2 = (x - 1)(x - 2)(x^2 + x - 1).$$

The roots of  $(x^2 + x - 1)$  are  $(-1 \pm \sqrt{5})/2$ , and so finally the four roots are

$$x = 1, 2, (-1 + \sqrt{5})/2, (-1 - \sqrt{5})/2.$$

2.7 Using the bisection method gives,

$n$	$x_n$		$f(x_n)$
1	$x_1 =$	1.500000000	-0.125000000
2	$x_2 =$	1.600000000	0.376000000
3	$x_3 = \frac{1}{2}(x_1 + x_2) =$	1.550000000	0.11888000
4	$x_4 = \frac{1}{2}(x_1 + x_3) =$	1.525000000	-0.00467200
5	$x_5 = \frac{1}{2}(x_3 + x_4) =$	1.537500000	0.05669000
6	$x_6 = \frac{1}{2}(x_4 + x_5) =$	1.531250000	0.02591100
7	$x_7 = \frac{1}{2}(x_4 + x_6) =$	1.528125000	0.01059000
8	$x_8 = \frac{1}{2}(x_4 + x_7) =$	1.526562500	0.00295500
9	$x_9 = \frac{1}{2}(x_4 + x_8) =$	1.525781250	-0.00086020
10	$x_{10} = \frac{1}{2}(x_8 + x_9) =$	1.526171875	0.00104700
11	$x_{11} = \frac{1}{2}(x_9 + x_{10}) =$	1.525976563	0.00009315

Thus the root correct to three decimal places is 1.526.

2.8 (a) Setting

$$\frac{2(x^2 - 9x + 11)}{(x - 2)(x - 3)(x + 4)} = \frac{A}{(x - 2)} + \frac{B}{(x - 3)} + \frac{C}{(x + 4)},$$

gives

$$2(x^2 - 9x + 11) = A(x - 3)(x + 4) + B(x - 2)(x + 4) + C(x - 2)(x - 3)$$

and using  $x = 2$ ,  $x = 3$  and  $x = -4$  in turn, yields  $A = 1$ ,  $B = -2$  and  $C = 3$ , so that

$$\frac{2(x^2 - 9x + 11)}{(x - 2)(x - 3)(x + 4)} = \frac{1}{(x - 2)} - \frac{2}{(x - 3)} + \frac{3}{(x + 4)}.$$

(b) Setting

$$\frac{7x^2 + 6x - 13}{(2x + 1)(x^2 + 2x - 4)} = \frac{Ax + B}{x^2 + 2x - 4} + \frac{C}{2x + 1},$$

gives

$$7x^2 + 6x - 13 = (Ax + B)(2x + 1) + C(x^2 + 2x - 4)$$

and equating coefficients of powers of  $x$  yields  $A = 2$ ,  $B = -1$  and  $C = 3$ , so that

$$\frac{7x^2 + 6x - 13}{(2x + 1)(x^2 + 2x - 4)} = \frac{2x - 1}{x^2 + 2x - 4} + \frac{3}{2x + 1}.$$

(c) Setting

$$\frac{2(3x^2 + 4x + 2)}{(x - 1)(2x + 1)^2} = \frac{A}{x - 1} + \frac{B}{2x + 1} + \frac{C}{(2x + 1)^2},$$

gives

$$2(3x^2 + 4x + 2) = A(2x + 1)^2 + B(2x + 1)(x - 1) + C(x - 1)$$

and equating coefficients of powers of  $x$  yields  $A = 2$ ,  $B = -1$  and  $C = -1$ , so that

$$\frac{2(3x^2 + 4x + 2)}{(x - 1)(2x + 1)^2} = \frac{2}{x - 1} - \frac{1}{2x + 1} - \frac{1}{(2x + 1)^2}.$$

2.9 (a) Performing a long division gives

$$\frac{x^3 - 2x^2 + 10}{(x - 1)(x + 2)} = (x - 3) + \frac{5x + 4}{(x - 1)(x + 2)}.$$

Then, setting

$$\frac{5x + 4}{(x - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 2}$$

gives

$$A(x + 2) + B(x - 1) = 5x + 4 \Rightarrow A = 3, B = 2$$

and so

$$\frac{x^3 - 2x^2 + 10}{(x-1)(x+2)} = (x-3) + \frac{3}{(x-1)} + \frac{2}{(x+2)}.$$

(b) Setting

$$\frac{3x^2 - 5x - 4}{(x+2)(3x^2 + x - 1)} = \frac{A}{(x+2)} + \frac{Bx + C}{(3x^2 + x - 1)},$$

gives

$$3x^2 - 5x - 4 = A(3x^2 + x - 1) + (x+2)(Bx + C),$$

and choosing  $x = -2$ , yields  $A = 2$ . Then

$$3x^2 - 5x - 4 = (6 + B)x^2 + (2B + C + 2)x + (2C - 2),$$

and equating coefficients of powers of  $x$ , gives  $B = -3$  and  $C = -1$ . So finally,

$$\frac{3x^2 - 5x - 4}{(x+2)(3x^2 + x - 1)} = \frac{2}{(x+2)} - \frac{3x+1}{(3x^2 + x - 1)}.$$

(c) Setting

$$\frac{3x^2 - x + 2}{(x-1)(x-3)^3} = \frac{A}{(x-1)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3},$$

gives

$$3x^2 - x + 2 = A(x-3)^3 + B(x-1)(x-3)^2 + C(x-1)(x-3) + D(x-1).$$

Letting  $x = 1$  yields  $A = -1/2$ , and letting  $x = 3$  yields  $D = 13$ . Then equating the coefficients of  $x^3$  gives  $B = 1/2$  and equating the constants on both sides gives  $C = 2$ . So, finally

$$\frac{3x^2 - x + 2}{(x-1)(x-3)^3} = -\frac{1}{2(x-1)} + \frac{1}{2(x-3)} + \frac{2}{(x-3)^2} + \frac{13}{(x-3)^3}.$$

2.10 (a) Using the double-angle formulas,

$$\cos 4\theta \equiv 2\cos^2 2\theta - 1 \quad \text{and} \quad \cos 2\theta \equiv 2\cos^2 \theta - 1,$$

so that

$$\cos 4\theta \equiv 2(2\cos^2 \theta - 1)^2 \equiv 8\cos^4 \theta - 8\cos^2 \theta + 1.$$

(b) Using the identities (2.36c) and (2.36d).

$$\sin(n\theta) + \sin[(n+4)\theta] \equiv 2\sin[(n+2)\theta]\cos(2\theta)$$

and

$$\cos(n\theta) + \cos[(n+4)\theta] \equiv 2\cos[(n+2)\theta]\cos(2\theta),$$

in the left-hand side of the identity gives

$$\frac{\sin[(n+2)\theta][1+2\cos(2\theta)]}{\cos[(n+2)\theta][1+2\cos(2\theta)]} \equiv \tan[(n+2)\theta].$$

(c)

$$\begin{aligned} \left(\frac{\sin 5\theta}{\sin \theta}\right)^2 - \left(\frac{\cos 5\theta}{\cos \theta}\right)^2 &= \left(\frac{\sin 5\theta}{\sin \theta} - \frac{\cos 5\theta}{\cos \theta}\right) \left(\frac{\sin 5\theta}{\sin \theta} + \frac{\cos 5\theta}{\cos \theta}\right) \\ &= \left(\frac{\sin 5\theta \cos \theta - \cos 5\theta \sin \theta}{\sin \theta \cos \theta}\right) \left(\frac{\sin 5\theta \cos \theta + \cos 5\theta \sin \theta}{\sin \theta \cos \theta}\right) \\ &= \frac{4 \sin 4\theta \sin 6\theta}{\sin 2\theta \sin 2\theta}. \end{aligned}$$

Then from (2.36c),  $\sin 6\theta$  may be written

$$\sin 6\theta = \sin 4\theta \cos 2\theta + \cos 4\theta \sin 2\theta,$$

and using the double-angle formulas (2.37a) on the right-hand side,

$$\sin 6\theta = 3 \sin 2\theta - 4 \sin^3 2\theta,$$

so that

$$\frac{4 \sin 4\theta \sin 6\theta}{\sin 2\theta \sin 2\theta} = \frac{8 \sin 2\theta \cos 2\theta (3 \sin 2\theta - 4 \sin^3 2\theta)}{\sin^2 2\theta},$$

and finally

$$\left(\frac{\sin 5\theta}{\sin \theta}\right)^2 - \left(\frac{\cos 5\theta}{\cos \theta}\right)^2 = 8 \cos 2\theta (3 - 4 \sin^2 2\theta) = 8 \cos 2\theta (4 \cos^2 2\theta - 1).$$

2.11 (a) Using the double-angle formula (2.37a), we have

$$2 \cos \theta \cos 2\theta + 2 \sin \theta \cos \theta = 2 \cos \theta (3 \cos^2 \theta - 1),$$

so the first solution is  $\cos \theta = 0$ , i.e.  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ . If  $\cos \theta \neq 0$ , we can divide by  $2 \cos \theta$  to give

$$\cos 2\theta + \sin \theta = 3 \cos^2 \theta - 1,$$

which again using the double-angle formula and writing everything in terms of  $\sin \theta$ , reduces to the quadratic  $\sin^2 \theta + \sin \theta - 1 = 0$  which leads directly to solutions  $\theta = 0.666$  and  $2.475$  radians.

(b) Combining the first and third terms using (2.36) gives

$$2 \sin 2\theta \cos \theta - 2 \sin \theta \cos \theta = 0,$$

so the first solution is

$$\cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}.$$

If  $\cos \theta \neq 0$ , then dividing gives

$$\sin \theta (2 \cos \theta - 1) = 0.$$

The two possibilities are

$$\sin \theta = 0 \Rightarrow \theta = \pi, \text{ or } \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

So finally

$$\theta = \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ and } \frac{5\pi}{3}.$$

2.12 We have

$$\sin k\theta - \sin \theta \equiv 2 \sin\left[\frac{1}{2}(k-1)\theta\right] \cos\left[\frac{1}{2}(k+1)\theta\right].$$

Thus one of the two brackets must be zero. The first bracket is zero if

$$\frac{1}{2}(k-1)\theta = \pm n\pi, \text{ i.e. } k\theta = \pm n\pi + \theta \Rightarrow \theta = \pm 2n\pi/(k-1),$$

for all integer  $n$  and  $k \neq 1$ . The second bracket is zero if

$$\frac{1}{2}(k+1)\theta = \pm \frac{1}{2}(2n+1)\pi, \text{ i.e. } k\theta = \pm(2n+1)\pi - \theta \Rightarrow \theta = \pm(2n+1)\pi/(k+1),$$

for all integer  $n$  and  $k \neq -1$ .

2.13 From the equation of the straight line  $x = (p - y \cos \theta)/\sin \theta$  and substituting into the equation for the hyperbola gives

$$\frac{p^2 - 2py \cos \theta + y^2 \cos^2 \theta}{a^2 \sin^2 \theta} - \frac{y^2}{b^2} = 1,$$

which is a quadratic in  $y$  of the form  $Ay^2 + By + C = 0$ , where

$$A = \left[ \frac{\cos^2 \theta}{a^2 \sin^2 \theta} - \frac{1}{b^2} \right], \quad B = - \left[ \frac{2p \cos \theta}{a^2 \sin^2 \theta} \right] \text{ and } C = \left[ \frac{p^2}{a^2 \sin^2 \theta} - 1 \right].$$

If the line is to be a tangent, then there can be only one solution of this quadratic, the condition for which is that  $B^2 = 4AC$ , i.e.

$$\left[ \frac{2p \cos \theta}{a^2 \sin^2 \theta} \right]^2 = 4 \left[ \frac{\cos^2 \theta}{a^2 \sin^2 \theta} - \frac{1}{b^2} \right] \left[ \frac{p^2}{a^2 \sin^2 \theta} - 1 \right],$$

which after simplifying gives  $a^2 \sin^2 \theta - b^2 \cos^2 \theta = p^2$ , as required.

The  $y$ -co-ordinate of the point of intersection is  $y = -B/2A$ , i.e.

$$y = \frac{2p \cos \theta}{2a^2 \sin^2 \theta} \cdot \frac{a^2 b^2 \sin^2 \theta}{b^2 \cos^2 \theta - a^2 \sin^2 \theta} = \frac{-b^2 \cos \theta}{p}$$

and hence

$$x = \frac{p - y \cos \theta}{\sin \theta} = \frac{a^2 \sin \theta}{p}.$$

2.14 Equation (2.37c) may be used on the left-hand side to give

$$\begin{aligned} \frac{1 + \sin \theta + \cos \theta}{1 + \sin \theta - \cos \theta} &\equiv \frac{1 + (2 \sin \theta / 2 \cos \theta / 2) + (2 \cos^2 \theta / 2 - 1)}{1 + (2 \sin \theta / 2 \cos \theta / 2) - (1 - 2 \sin^2 \theta / 2)} \\ &\equiv \frac{\cos \theta / 2}{\sin \theta / 2} \equiv \frac{2 \cos^2 \theta / 2}{2 \sin \theta / 2 \cos \theta / 2} \equiv \frac{1 + \cos \theta}{\sin \theta}. \end{aligned}$$

2.15 From the sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \sin A = \frac{a \sin B}{b}.$$

So  $A = \arcsin[5 \sin(0.5)/4] = 0.643 \text{ rad} = 36.84^\circ$  and hence  $C = 1.999 \text{ rad} = 114.51^\circ$ .

Again using the sine rule,  $c = b \sin C / \sin B = 7.59 \text{ cm}$ .

2.16 The lengths of the sides are:  $a = BC = \sqrt{(5-7)^2 + (6-2)^2} = \sqrt{20}$  and likewise  $b = AC = \sqrt{37}$  and  $c = AB = \sqrt{25}$ . Then using the cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{21}{5\sqrt{37}},$$

giving  $A = 0.809 \text{ rad} = 46.35^\circ$ . In a similar way,  $B = 1.391 \text{ rad} = 79.70^\circ$  and  $C = 0.942 \text{ rad} = 53.97^\circ$ .

2.17 For  $n = 0$  the results are trivial. So we just need to show that if both are true for any  $n \geq 0$  they are true for  $m = n + 1$ . Suppose both results are true for  $n$ . Then

$$\begin{aligned} \sin[(2m+1)\theta] &= \sin[(2n+1)\theta + 2\theta] \\ &= \sin[(2n+1)\theta] \cos 2\theta + \cos[(2n+1)\theta] \sin 2\theta \\ &= \sin[(2n+1)\theta][1 - 2\sin^2 \theta] + \left[ \frac{\cos[(2n+1)\theta]}{\cos \theta} \right] 2\sin \theta \cos^2 \theta, \end{aligned}$$

which is a polynomial in  $\sin \theta$  since both  $\sin[(2n+1)\theta]$  and  $\cos[(2n+1)\theta]/\cos \theta$  are polynomials and  $\cos^2 \theta = 1 - \sin^2 \theta$ . Similarly,