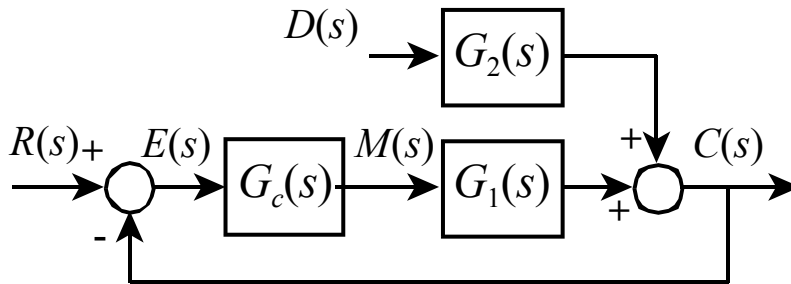


Problem 6-1. Second-order loop with proportional controller.



$$G_1(s) = \frac{K}{(\tau_1 \cdot s + 1) \cdot (\tau_2 \cdot s + 1)} \quad G_c(s) = K_c$$

Problem parameters: $K := 0.10 \frac{\%TO}{\%CO}$ $\tau_1 := 1 \text{ min}$ $\tau_2 := 0.8 \text{ min}$

(a) Closed loop transfer function and characteristic equation of the loop.

$$\frac{C(s)}{R(s)} = \frac{K_c \cdot \frac{K}{(\tau_1 \cdot s + 1) \cdot (\tau_2 \cdot s + 1)}}{1 + K_c \cdot \frac{K}{(\tau_1 \cdot s + 1) \cdot (\tau_2 \cdot s + 1)}} = \frac{K_c \cdot K}{(\tau_1 \cdot s + 1) \cdot (\tau_2 \cdot s + 1) + K_c \cdot K}$$

Characteristic equation: $\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2)s + 1 + K_c \cdot K = 0$

Closed-loop transfer function:
$$\frac{C(s)}{R(s)} = \frac{0.1K_c}{0.8s^2 + 1.8s + 1 + 0.1K_c}$$

Characteristic equation:
$$0.8s^2 + 1.8s + 1 + 0.1K_c = 0$$

(b) Values of the controller gain for which the response is over-damped, critically damped, and under-damped

Roots of the characteristic equation:

$$r_1 = \frac{-1.8 + \sqrt{1.8^2 - 4 \cdot 0.8 \cdot (1 + 0.1K_c)}}{2 \cdot 0.8} = \frac{-1.8}{1.6} + \sqrt{\left(\frac{1.8}{1.6}\right)^2 - \frac{1 + 0.1K_c}{0.8}}$$

The response is critically damped when the term in the radical is zero:

$$\left(\frac{1.8}{1.6}\right)^2 - \frac{1 + 0.1K_c}{0.8} = 0$$

Critically damped: $K_{ccd} := \frac{1}{0.1} \left[0.8 \left(\frac{1.8}{1.6}\right)^2 - 1 \right]$ $K_{ccd} = 0.125 \frac{\%CO}{\%TO}$

Over-damped (real roots): $K_c < 0.125 \frac{\%CO}{\%TO}$ Under-damped: $K_c > 0.125 \frac{\%CO}{\%TO}$

The loop cannot be unstable for positive gain because,

- for real roots the radical cannot be greater than the negative term, so both roots are negative
- for complex conjugate roots the real part is always negative, $-1.8/1.6$, or $-(\tau_1 + \tau_2)/2\tau_1\tau_2$

This is true for all positive values of the time constants and the product $K_c K$.

(c) Equivalent time constants for different values of the gain:

$K_c := 0.1 \frac{\%CO}{\%TO}$ (over-damped, two equivalent time constants)

$$\tau_{e1} = \frac{-1}{r_1} \quad \tau_{e1} := \frac{2 \cdot \tau_1 \cdot \tau_2}{(\tau_1 + \tau_2) - \sqrt{(\tau_1 + \tau_2)^2 - 4 \cdot \tau_1 \cdot \tau_2 \cdot (1 + K_c \cdot K)}} \quad \tau_{e1} = 0.935 \text{ min}$$

$$\tau_{e2} = \frac{-1}{r_2} \quad \tau_{e2} := \frac{2 \cdot \tau_1 \cdot \tau_2}{(\tau_1 + \tau_2) + \sqrt{(\tau_1 + \tau_2)^2 - 4 \cdot \tau_1 \cdot \tau_2 \cdot (1 + K_c \cdot K)}} \quad \tau_{e2} = 0.847 \text{ min}$$

$K_c := 0.125 \frac{\%CO}{\%TO}$ (critically damped, two equal real time constants)

$$\tau_{e1} = \frac{-1}{r_1} \quad \tau_{e1} := \frac{2 \cdot \tau_1 \cdot \tau_2}{(\tau_1 + \tau_2) - \sqrt{(\tau_1 + \tau_2)^2 - 4 \cdot \tau_1 \cdot \tau_2 \cdot (1 + K_c \cdot K)}} \quad \tau_{e1} = 0.889 \text{ min}$$

$$\tau_{e2} = \frac{-1}{r_2} \quad \tau_{e2} := \frac{2 \cdot \tau_1 \cdot \tau_2}{(\tau_1 + \tau_2) + \sqrt{(\tau_1 + \tau_2)^2 - 4 \cdot \tau_1 \cdot \tau_2 \cdot (1 + K_c \cdot K)}} \quad \tau_{e2} = 0.889 \text{ min}$$

$K_c := 0.2 \frac{\%CO}{\%TO}$ (under-damped, time constant and damping ratio)

$$\tau^2 s^2 + 2\zeta \cdot \tau \cdot s + 1 = \frac{\tau_1 \cdot \tau_2}{1 + K_c \cdot K} s^2 + \frac{\tau_1 + \tau_2}{1 + K_c \cdot K} s + 1$$

Match coefficients: $\tau := \sqrt{\frac{\tau_1 \cdot \tau_2}{1 + K_c \cdot K}}$ $\zeta := \frac{\tau_1 + \tau_2}{2 \cdot \tau \cdot (1 + K_c \cdot K)}$ $\tau = 0.886 \text{ min}$ $\zeta = 0.996$

(d) Steady-state offset for a unit step change in set point.

Final value theorem: $\lim_{t \rightarrow \infty} Y(t) = \lim_{s \rightarrow 0} s \cdot Y(s)$ $R(s) = \frac{1}{s}$ (Table 2-1.1)

$$K_c := 0.1 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 9.9009900990099009901 \cdot 10^{-3}$$

$$\text{offset} := (1 - 0.0099)\%TO$$

$$\boxed{\text{offset} = 0.99\%TO}$$

$$K_c := 0.125 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 1.2345679012345679012 \cdot 10^{-2}$$

$$\text{offset} := (1 - 0.01235)\%TO$$

$$\boxed{\text{offset} = 0.988\%TO}$$

$$K_c := 0.2 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot K}{\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2) \cdot s + 1 + K_c \cdot K} \frac{1}{s} \rightarrow 1.9607843137254901961 \cdot 10^{-2}$$

$$\text{offset} := (1 - 0.01961)\%TO$$

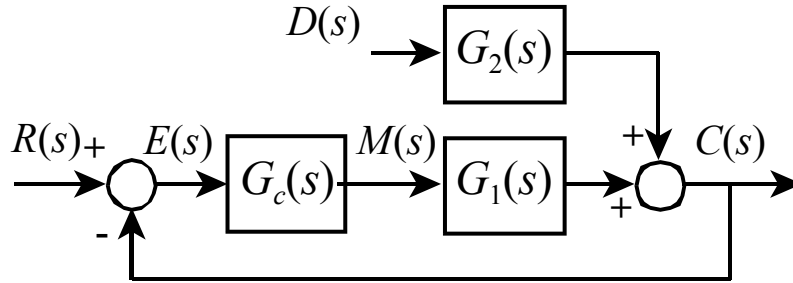
$$\boxed{\text{offset} = 0.98\%TO}$$

These are very large offsets because the loop gains are so small.

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Problem 6-2. Inverse-response second-order system with proportional controller.



$$G_1(s) = \frac{6(1-s)}{(s+1) \cdot (0.5s+1)} \frac{\%TO}{\%CO} \quad G_c(s) = K_c \cdot \frac{\%CO}{\%TO}$$

(a) Closed-loop transfer function and characteristic equation of the loop.

Closed-loop transfer function:
$$\frac{C(s)}{R(s)} = \frac{K_c \cdot 6(1-s)}{(s+1) \cdot (0.5s+1) + K_c \cdot 6(1-s)}$$

Characteristic equation:
$$0.5 \cdot s^2 + (1.5 - 6K_c)s + 1 + 6K_c = 0$$

(b) Values of the gain for which the response is over-, critically, and under-damped

Roots:

$$r_1 = \frac{-(1.5 - 6K_c) + \sqrt{(1.5 - 6K_c)^2 - 4 \cdot 0.5 \cdot (1 + 6K_c)}}{2 \cdot 0.5} = -1.5 + 6K_c + \sqrt{0.25 - 30K_c + 36K_c^2}$$

The response is critically damped when the term in the radical is zero:

$$0.25 - 30K_c + 36K_c^2 = 0 \quad K_c := \frac{30 + \sqrt{30^2 - 4 \cdot 0.25 \cdot 36}}{2 \cdot 36} \quad K_c = 0.82491 \frac{\%CO}{\%TO}$$

$$K_c := \frac{30 - \sqrt{30^2 - 4 \cdot 0.25 \cdot 36}}{2 \cdot 36} \quad K_c = 0.00842 \frac{\%CO}{\%TO}$$

Over-damped (two real roots): $K_c < 0.00842 \frac{\%CO}{\%TO}$ and $K_c > 0.825 \frac{\%CO}{\%TO}$

Under-damped (complex conjugate roots): $0.00842 \frac{\%CO}{\%TO} < K_c < 0.825 \frac{\%CO}{\%TO}$

The response is unstable when $K_c > 0.25 \frac{\%CO}{\%TO}$ (one real root is positive or the real part of the complex roots is positive)

(c) Effective time constants or time constant and damping ratio for various values of the gain:

$$\frac{0.5}{1 + 6K_c} s^2 + \frac{1.5 - 6K_c}{1 + 6K_c} s + 1 = \tau^2 s^2 + 2\zeta \cdot \tau \cdot s + 1$$

$K_c := 0.1 \frac{\%CO}{\%TO}$	$\tau := \sqrt{\frac{0.5 \text{min}^2}{1 + 6K_c}}$	$\zeta := \frac{(1.5 - 6K_c) \text{min}}{2 \cdot \tau \cdot (1 + 6K_c)}$	$\tau = 0.559 \text{ min}$	$\zeta = 0.503$
$K_c := 0.125 \frac{\%CO}{\%TO}$	$\tau := \sqrt{\frac{0.5 \text{min}^2}{1 + 6K_c}}$	$\zeta := \frac{(1.5 - 6K_c) \text{min}}{2 \cdot \tau \cdot (1 + 6K_c)}$	$\tau = 0.535 \text{ min}$	$\zeta = 0.401$
$K_c := 0.2 \frac{\%CO}{\%TO}$	$\tau := \sqrt{\frac{0.5 \text{min}^2}{1 + 6K_c}}$	$\zeta := \frac{(1.5 - 6K_c) \text{min}}{2 \cdot \tau \cdot (1 + 6K_c)}$	$\tau = 0.477 \text{ min}$	$\zeta = 0.143$
$K_c := 0.3 \frac{\%CO}{\%TO}$	$\tau := \sqrt{\frac{0.5 \text{min}^2}{1 + 6K_c}}$	$\zeta := \frac{(1.5 - 6K_c) \text{min}}{2 \cdot \tau \cdot (1 + 6K_c)}$	$\tau = 0.423 \text{ min}$	$\zeta = -0.127$ (unstable)

Try values that result in equivalent time constants:

$K_c := 0.005 \frac{\%CO}{\%TO}$	$\tau_{e1} := \frac{1 \text{min}}{1.5 - 6 \cdot K_c - \sqrt{0.25 - 30K_c + 36K_c^2}}$	$\tau_{e1} = 0.868 \text{ min}$
	$\tau_{e2} := \frac{1 \text{min}}{1.5 - 6 \cdot K_c + \sqrt{0.25 - 30K_c + 36K_c^2}}$	$\tau_{e2} = 0.559 \text{ min}$
$K_c := 1 \frac{\%CO}{\%TO}$	$\tau_{e1} := \frac{1 \text{min}}{1.5 - 6 \cdot K_c - \sqrt{0.25 - 30K_c + 36K_c^2}}$	$\tau_{e1} = -0.143 \text{ min}$ (unstable)
	$\tau_{e2} := \frac{1 \text{min}}{1.5 - 6 \cdot K_c + \sqrt{0.25 - 30K_c + 36K_c^2}}$	$\tau_{e2} = -0.5 \text{ min}$

(d) Offset for various values of the gain and a unit step change in set point.

$$K_c := 0.10 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot 6 \cdot (1 - s)}{0.5s^2 + (1.5 - 6 \cdot K_c)s + 1 + 6K_c} \cdot \frac{1}{s} \rightarrow .37500000000000000000$$

offset := 1 - 0.375

offset = 0.625 $\frac{\%CO}{\%TO}$

$$K_c := 0.125 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot 6 \cdot (1 - s)}{0.5s^2 + (1.5 - 6 \cdot K_c)s + 1 + 6K_c} \frac{1}{s} \rightarrow .42857142857142857143$$

$$\text{offset} := 1 - 0.429$$

offset = 0.571 $\frac{\%CO}{\%TO}$

$$K_c := 0.20 \frac{\%CO}{\%TO} \quad \lim_{s \rightarrow 0} s \cdot \frac{K_c \cdot 6 \cdot (1 - s)}{0.5s^2 + (1.5 - 6 \cdot K_c)s + 1 + 6K_c} \frac{1}{s} \rightarrow .54545454545454545455$$

$$\text{offset} := 1 - 0.545$$

offset = 0.455 $\frac{\%CO}{\%TO}$

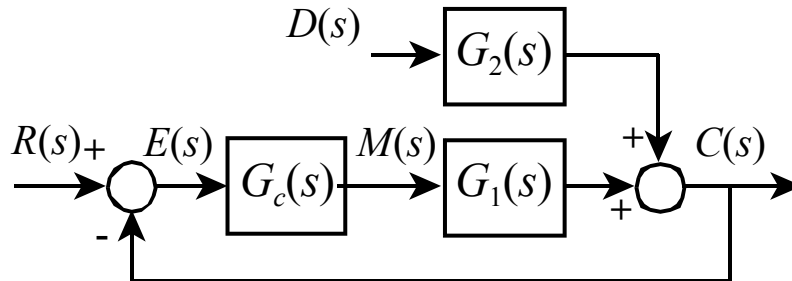
The offsets are high because the gains are small. Of course, since for gains greater than 0.25%CO/%TO the loop is unstable, offsets can only be high with a proportional controller.

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$$K := 1.8R$$

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Problem 6-3. First-order process and proportional-integral controller.



$$G_1(s) = \frac{K}{\tau \cdot s + 1} \quad G_c(s) = K_c \cdot \left(1 + \frac{1}{\tau_I s} \right)$$

To work in dimensionless units, t/τ , set: $\tau := 1$

(a) Closed-loop transfer function and characteristic equation of the loop. Offset.

Closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{KK_c \cdot (\tau_I s + 1)}{\tau_I s \cdot (\tau \cdot s + 1) + KK_c \cdot (\tau_I s + 1)}$$

Characteristic equation:

$$\tau_I \tau \cdot s^2 + (1 + KK_c) \tau_I s + KK_c = 0$$

Offset: the steady state gain is:

$$\lim_{s \rightarrow 0} \frac{KK_c \cdot (\tau_I s + 1)}{\tau_I \tau \cdot s^2 + (1 + KK_c) \tau_I s + KK_c} = \frac{KK_c}{KK_c} = 1$$

(no offset)

(b) Is there an ultimate gain for this loop?

Substitute $s = i\omega$:

$$-\tau_I \tau \cdot \omega_u^2 + KK_{cu} + i(1 + KK_{cu}) \omega_u = 0 \quad \omega_u := 0 \quad KK_{cu} := 0$$

No, there is no ultimate gain. This result just means that a negative loop gain will make the loop unstable. Another way to show it is to determine the roots of the characteristic equation:

$$r_1 = \frac{-(1 + KK_c) \cdot \tau_I + \sqrt{(1 + KK_c)^2 \cdot \tau_I^2 - 4 \cdot \tau_I \tau \cdot KK_c}}{2 \tau_I \tau}$$

The real root cannot be negative for any positive value of the loop gain KK_c because the radical is always smaller than the negative term. Also, for complex conjugate roots, the real part is always

negative:

$$\text{Real} = \frac{-(1 + KK_c)}{2 \cdot \tau} < 0$$

(c) Response of the loop to a step change in set point for $\tau_1 = \tau$ as the gain varies from 0 to infinity.

$$R(s) = \frac{1}{s} \quad (\text{Table 2-1.1}) \quad Y(s) = \frac{KK_c \cdot (\tau \cdot s + 1)}{\tau \cdot s \cdot (\tau \cdot s + 1) + KK_c \cdot (\tau \cdot s + 1)} \frac{1}{s} = \frac{KK_c}{\tau \cdot s + KK_c} \frac{1}{s}$$

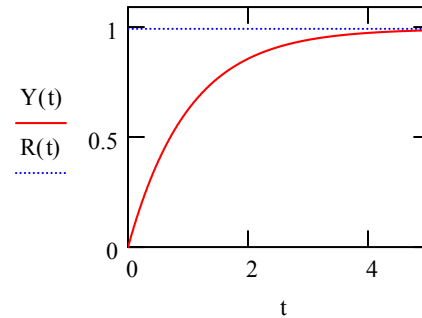
Let $\tau_c = \frac{\tau}{KK_c}$

$$Y(s) = \frac{1}{\tau_c \cdot s + 1} \frac{1}{s} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau_c}} \quad u(t) := \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Invert using Table 2-1.1:

$$Y(t) := u(t) - e^{-\frac{t}{\tau_c}} \quad R(t) := u(t) \quad \tau_c := 1$$

As KK_c increases τ_c decreases and the response is faster.

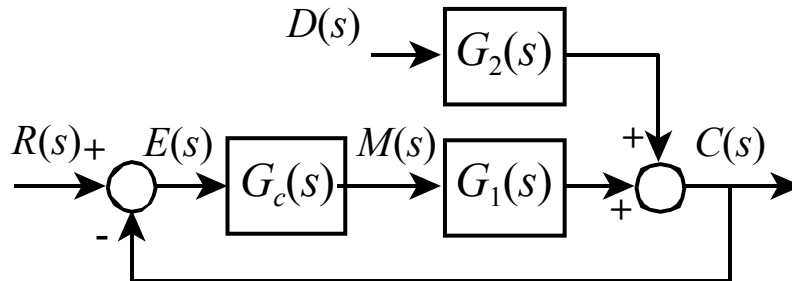


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$$R := \frac{K}{1.8}$$

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Problem 6-4. Second-order process with pure integral controller.



$$G_1(s) = \frac{K}{(\tau_1 \cdot s + 1) \cdot (\tau_2 \cdot s + 1)} \quad G_c(s) = \frac{K_I}{s}$$

(a) Ultimate gain and period with the parameters of Problem 6-1:

$$K := 0.1 \frac{\%CO}{\%TO} \quad \tau_1 := 1 \text{ min} \quad \tau_2 := 0.8 \text{ min}$$

Characteristic equation: $\tau_1 \cdot \tau_2 \cdot s^2 + (\tau_1 + \tau_2)s + 1 + \frac{K \cdot K_I}{s} = 0$

$$\tau_1 \cdot \tau_2 s^3 + (\tau_1 + \tau_2)s^2 + s + KK_I = 0$$

Substitute $s = i\omega_u$: $-\tau_1 \cdot \tau_2 \cdot i \cdot \omega_u^3 - (\tau_1 + \tau_2)\omega_u^2 + i \cdot \omega_u + KK_{Iu} = 0 + 0i$

$$-(\tau_1 + \tau_2)\omega_u^2 + KK_{Iu} = 0 \quad -\tau_1 \cdot \tau_2 \omega_u^3 + \omega_u = 0$$

$$\omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}} \quad K_{Iu} := \frac{(\tau_1 + \tau_2) \cdot \omega_u^2}{K} \quad \omega_u = 1.118 \text{ min}^{-1} \quad K_{Iu} = 22.5 \frac{\%CO}{\%TO \cdot \text{min}}$$

$$T_u := \frac{2\pi}{\omega_u} \quad T_u = 5.62 \text{ min}$$

(b) Ultimate gain and period for other values of the smaller time constant:

$$\tau_2 := 0.1 \text{ min} \quad \omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}} \quad K_{Iu} := \frac{(\tau_1 + \tau_2) \cdot \omega_u^2}{K} \quad \omega_u = 3.162 \text{ min}^{-1} \quad K_{Iu} = 110 \frac{\%CO}{\%TO \cdot \text{min}}$$

$$T_u := 2 \frac{\pi}{\omega_u} \quad T_u = 1.987 \text{ min}$$

$$\tau_2 := 2 \text{ min} \quad \omega_u := \sqrt{\frac{1}{\tau_1 \cdot \tau_2}} \quad K_{Iu} := \frac{(\tau_1 + \tau_2) \cdot \omega_u^2}{K} \quad \omega_u = 0.707 \text{ min}^{-1} \quad \boxed{K_{Iu} = 15 \frac{\%CO}{\%TO \cdot \text{min}}}$$

$$T_u := \frac{2\pi}{\omega_u} \quad \boxed{T_u = 8.886 \text{ min}}$$

Reducing the non-dominant time constant increases the ultimate gain and reduces the ultimate period, as expected. When τ_2 is increased to 2 min, it becomes the dominant time constant and the ultimate gain should be higher than for part (a). However, in this case K_I has units of rate and, since the loop is slower, it results in a smaller ultimate gain.

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